ON THE PLANARITY AND HAMILTONICITY OF HANOI GRAPHS

Katherine Rock
Advisor: Dr. John Caughman

Fariborz Maseeh Department of Mathematics and Statistics
Portland State University

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The Tower of Hanoi Puzzles

Édouard Lucas

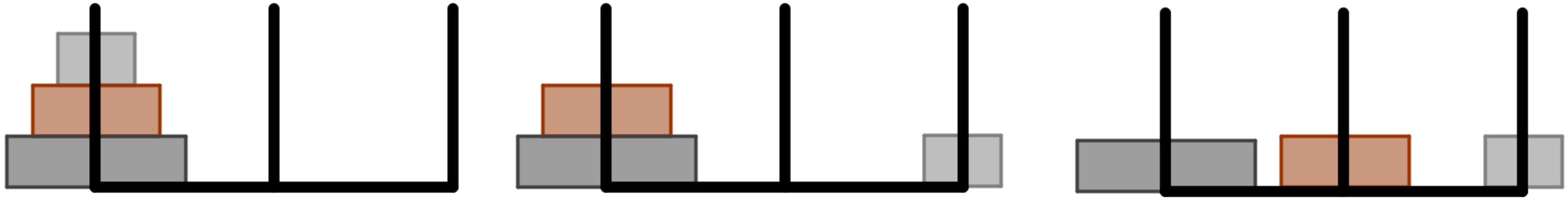
1842-1891

The original Tower of Hanoi puzzle

1883

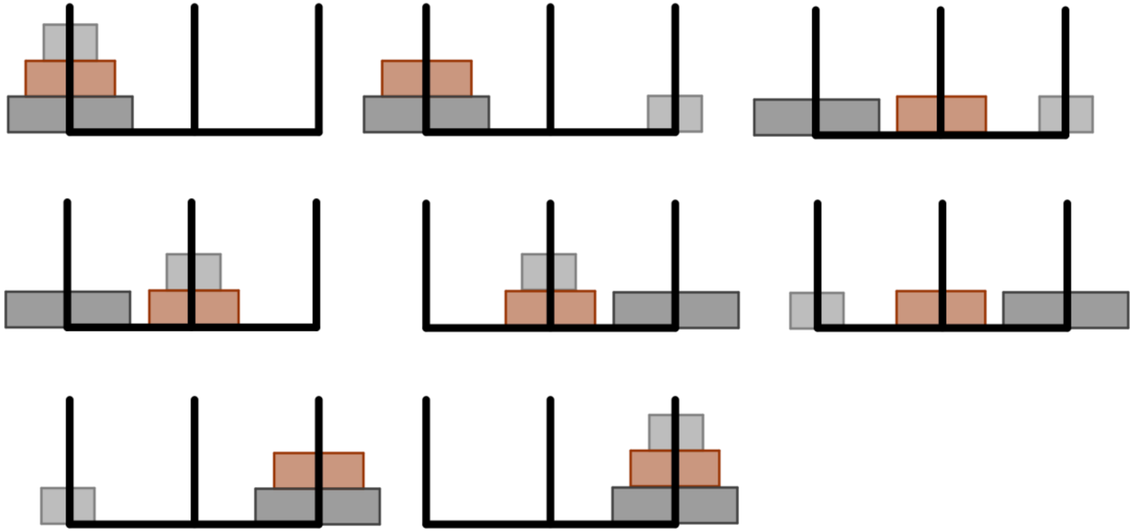
The Tower of Hanoi Puzzles

- n discs arranged on $3 + m$ vertical pegs, with $n, m \in \mathbb{Z}^{\geq 0}$.
- Each disc is a different size.
- *Regular state:* If multiple discs are on the same peg, they are arranged in decreasing size from bottom to top.
- *Perfect state:* A regular state in which all discs are on the same peg.



The Tower of Hanoi Puzzles

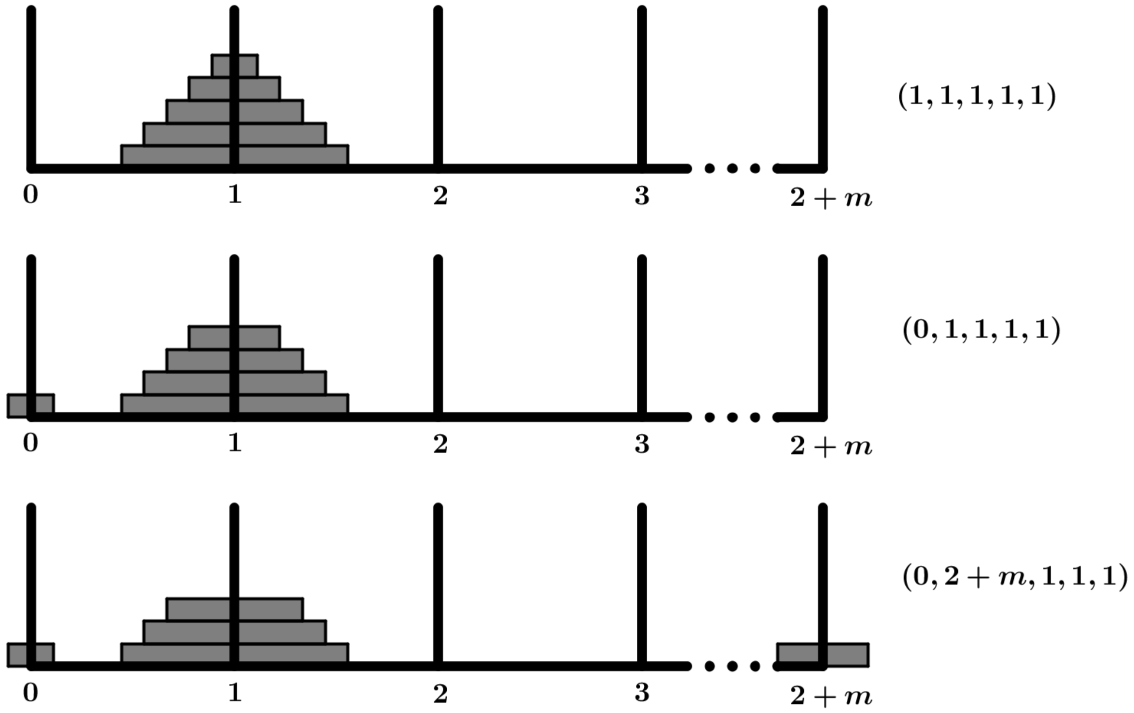
- *Object:* To move from one perfect state to another by moving one disc at a time from the topmost position on one peg to the topmost position on another peg.
- *Divine rule:* No larger disc may be placed on top of any smaller disc.



Hanoi Graphs

- The Hanoi graph H_m^n corresponds to the Tower of Hanoi puzzle with $3 + m$ pegs and n discs.
- Label the pegs $0,1, \ldots, 2 + m$ and let x_i be the position of the disc with radius i, for each $i = 1,2, \ldots, n$.
- Then each regular state in the puzzle is represented by vertex in the graph, labeled with an n-tuple $(x_1, x_2, \ldots, x_n)$, where each $x_i \in \{0,1, \ldots, 2 + m\}$.
- The edges of H_m^n are all the possible legal moves of the discs. Two vertices are adjacent if and only if their corresponding states can be achieved from one another through a legal move of exactly one disc.

Example: H_m^5



In the graph H_m^5,

$(1,1,1,1,1) \sim (0,1,1,1,1)$ and $(0,1,1,1,1) \sim (0,2+m,1,1,1)$,

but $(1,1,1,1,1) \nsim (0,2+m,1,1,1)$.

Hanoi Graphs

Definition

Let $n, m \in \mathbb{Z}$, with $n > 0$ and $m \geq 0$.

The Hanoi graph H_m^n is the graph with vertex set $V(H_m^n)$ given by

$$V(H_m^n) = \{(x_1, x_2, \ldots, x_n) | 0 \leq x_i \leq 2 + m, x_i \in \mathbb{Z}\}$$

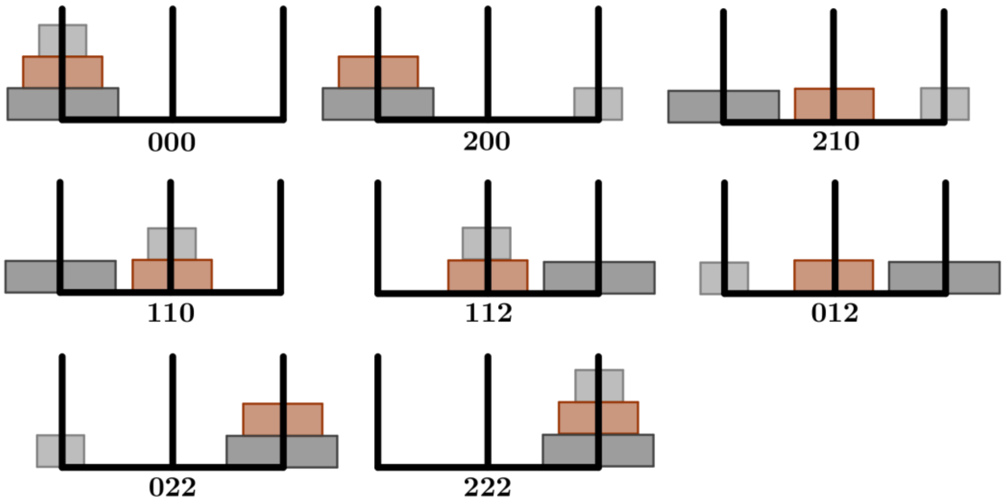
and where $(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n)$ if and only if there exists $i \in \{1,2, \ldots, n\}$ such that

i. $x_i \neq y_i$,

ii. $x_j = y_j$ for all $i \neq j$, and

iii. $\{x_i, y_i\} \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$.

Example: H_0^3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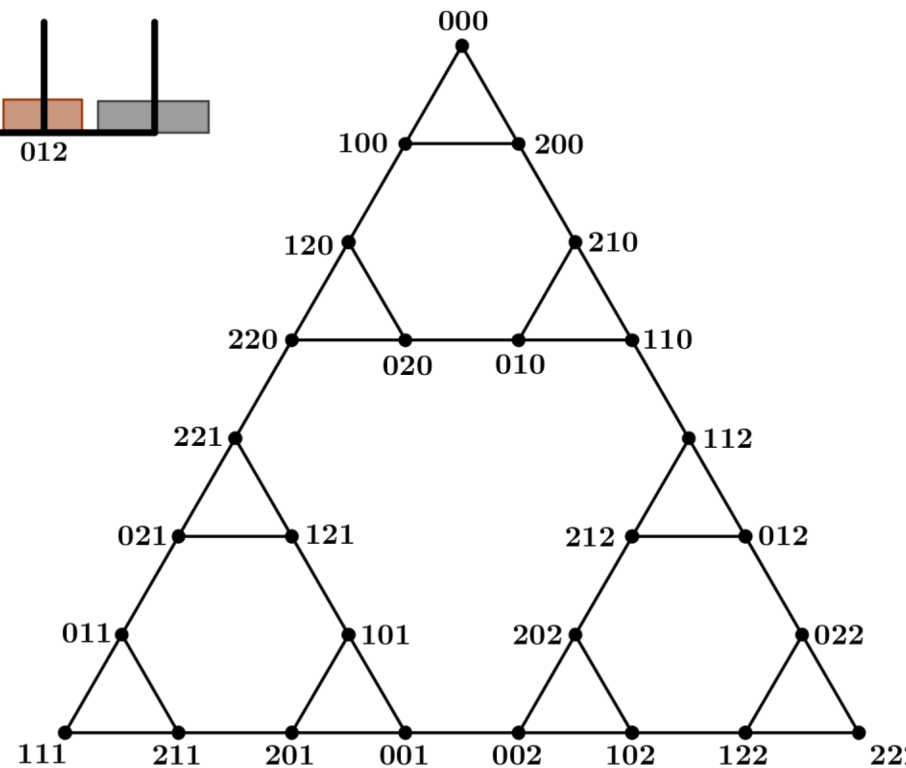
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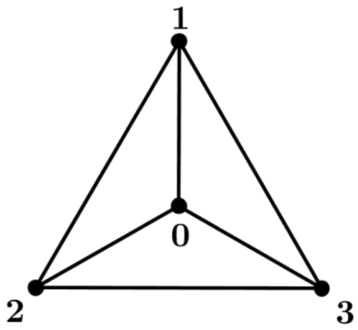
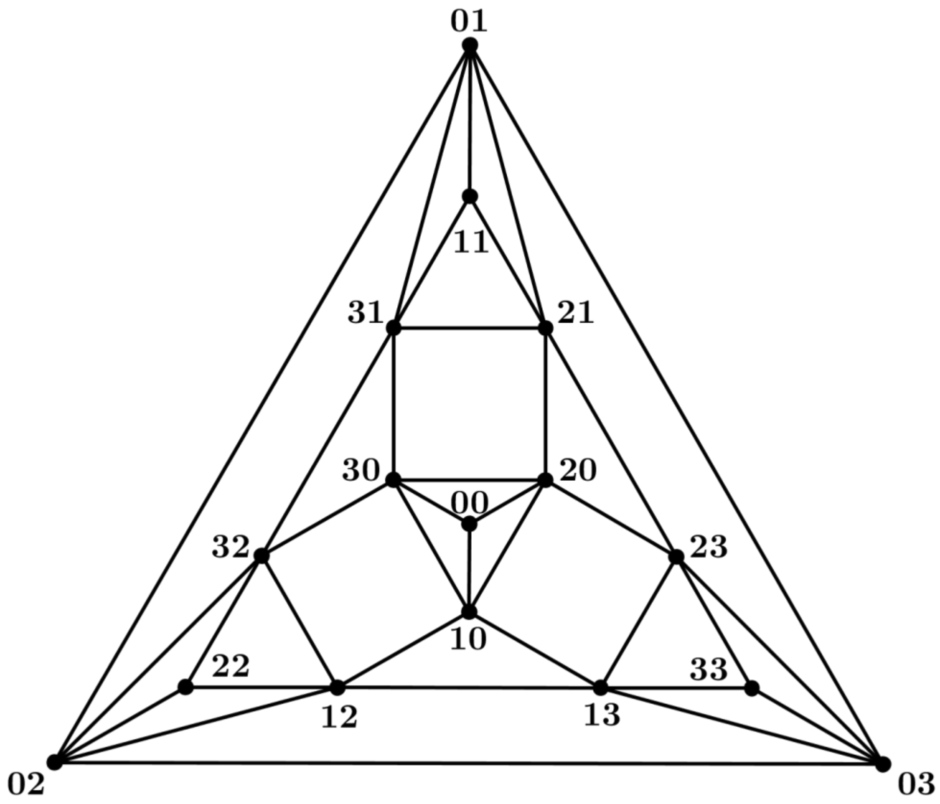
Example: H_1^1 & H_1^2





Outline

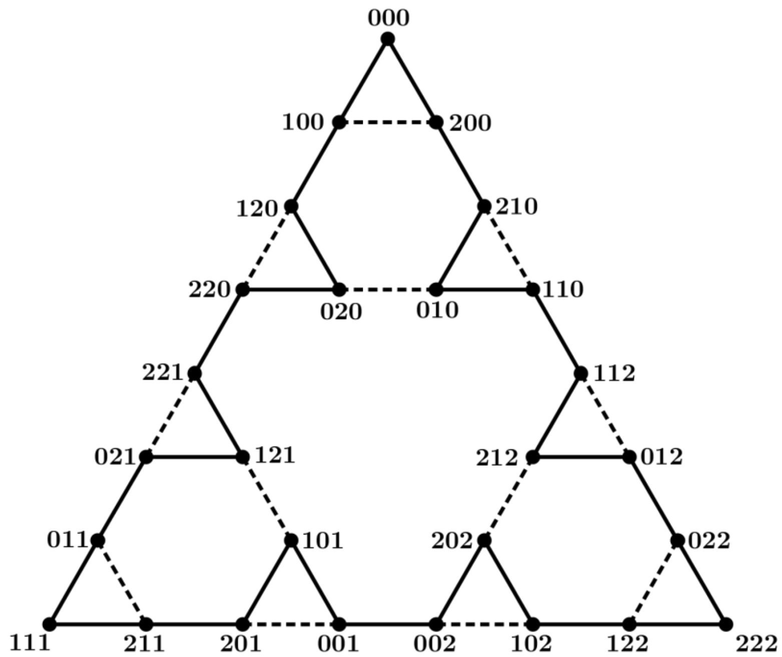
- Introduction (done)
- Hamiltonian graphs
- Hamiltonicity of H_m^n
- Planar graphs
- Planarity of Hanoi graphs

Hamiltonian Graphs

Definition

A graph G is called hamiltonian if it contains a cycle that is a spanning subgraph of G.

Example: H_0^3



Hamiltonicity of H_m^n

Lemma 1

Let s_1, s_2, s_3, and s_4 be perfect states in H_m^n, with $\mathrm{s}_1 \neq s_2$ and $s_3 \neq s_4$.

Then there exists an automorphism $f \in Aut(H_m^n)$ such that $f(s_1) = s_3$ and $f(s_2) = s_4$.

Hamiltonicity of H_m^n

Theorem 1

Every Hanoi graph is hamiltonian.

Proof: Fix any $m \in \mathbb{Z}^{\geq 0}$.

The proof consists of two parts.

- Part I: We will show by induction on n that there exists a hamiltonian path in H_m^n beginning and ending with vertices that correspond to distinct perfect states.
- Part II: We will use the result of Part I to construct a hamiltonian cycle in H_m^{n+1}.

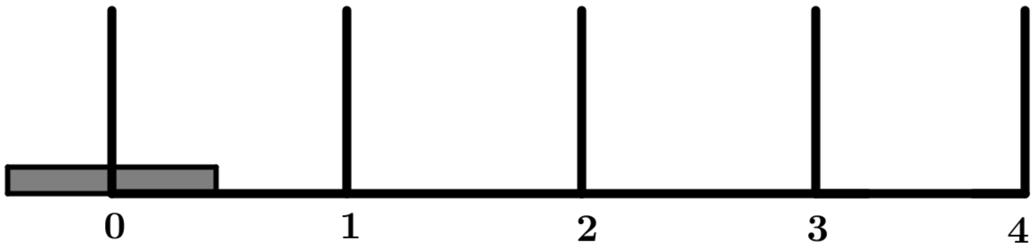
Theorem 1, Part I

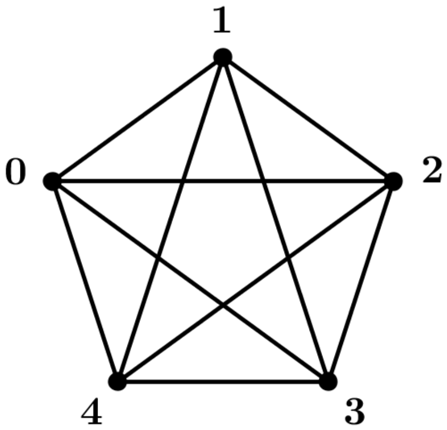
Base Case:

Let $n = 1$.

The Hanoi graph H_m^1 is isomorphic to the complete graph on $3 + m$ vertices, which is hamiltonian, and so contains a hamiltonian path.

Example: H_2^1



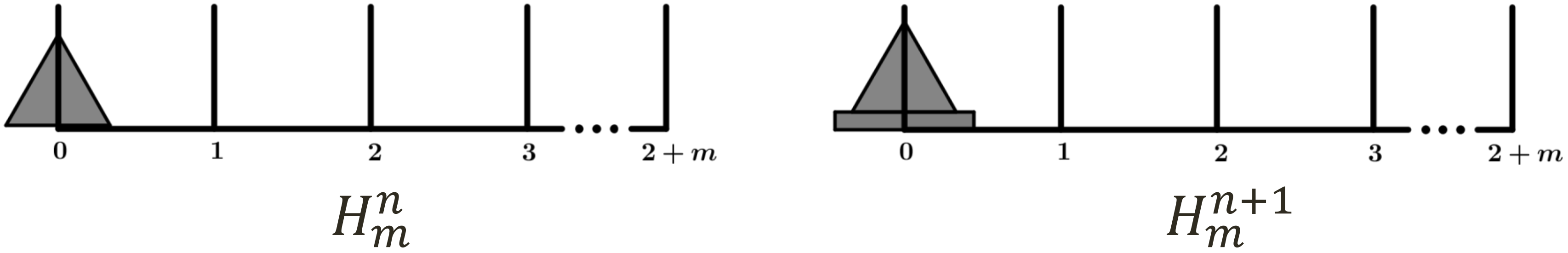


Theorem 1, Part I

Induction Hypothesis:

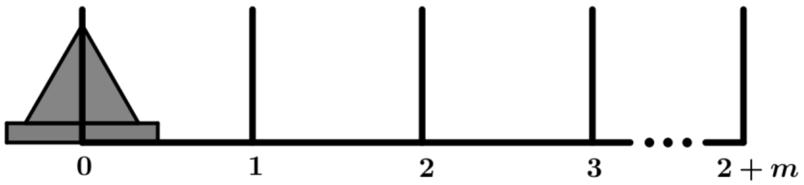
Fix any $n \geq 1$ and suppose H_m^n has a hamiltonian path beginning and ending with vertices that correspond to distinct perfect states.

H_m^{n+1} corresponds to the puzzle obtained by adding a disc with radius $n+1$ to the Tower of Hanoi puzzle that correspond to H_m^n.



Theorem 1, Part I

Without loss of generality, suppose all discs begin on peg 0.



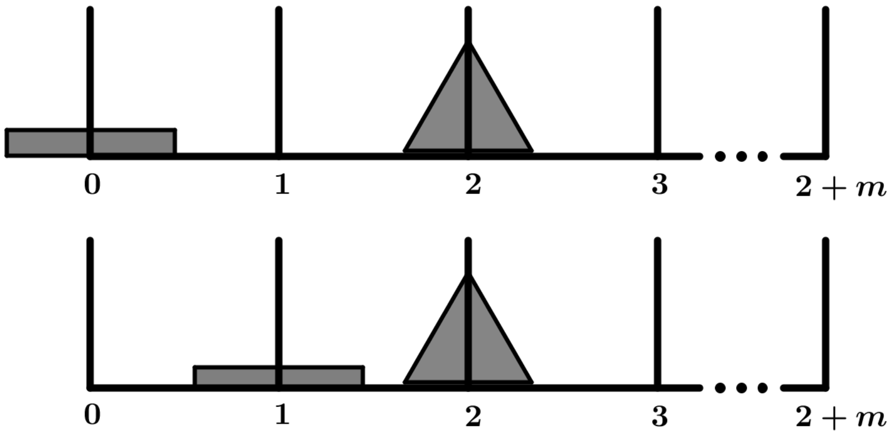
By the induction hypothesis, there is a hamiltonian path between distinct perfect states in H_m^n.

By Lemma 1, perfect states are isomorphic, so there is a hamiltonian path between any two distinct perfect states.

We can move disc $n+1$ stepwise through every peg from 0 to $2+m$ in the following way.

Theorem 1, Part I

Before each step moving disc $n + 1$, we perform a hamiltonian path transferring the n-tower of discs to a peg allowing disc $n + 1$ to move.

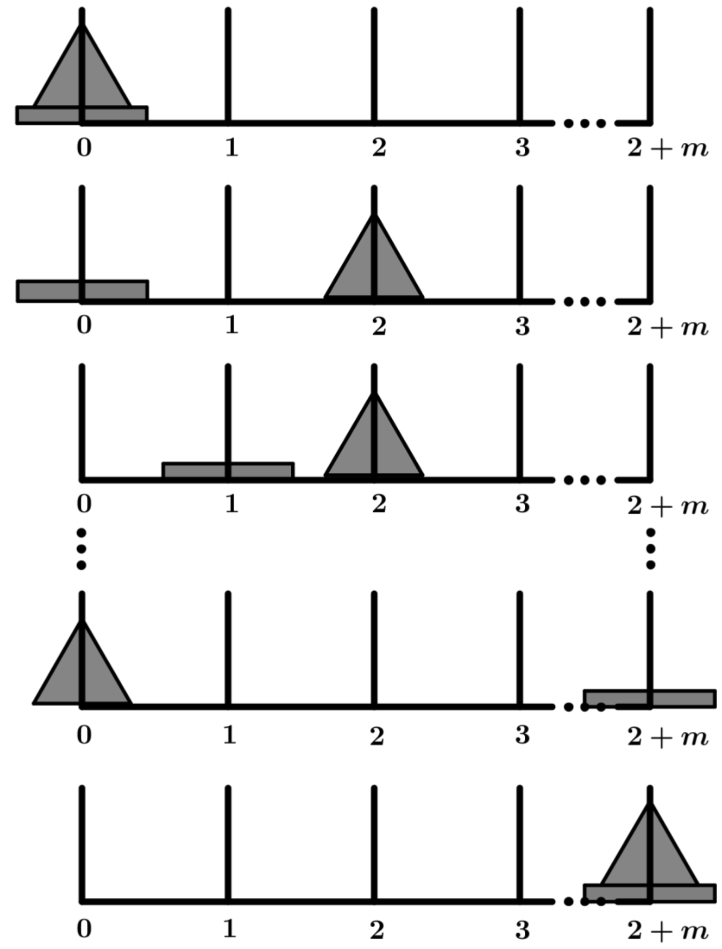


In general, before moving disc $n + 1$ from peg i to peg $i + 1$, we first move the n-tower to peg $i + 2 (\text{mod } 3 + m)$.

Theorem 1, Part I

After the last move of disc $n+1$ to peg $2+m$, the n-tower can be transferred to peg $2+m$ as well, again through a hamiltonian path in H_m^n.

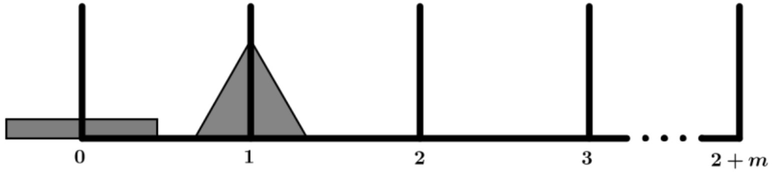
During this process, every possible state of all $n+1$ discs is achieved exactly once, completing a hamiltonian path in H_m^{n+1}.



Theorem 1, Part II

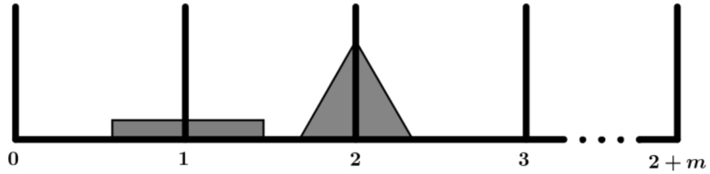
We now construct a hamiltonian cycle in H_m^{n+1}.

Without loss of generality, let the initial vertex in the cycle be $(1,1,\ldots,1,0) \in V(H_m^{n+1})$.



By Part I, we can transfer the n-tower of smaller discs from peg 1 to peg 2 through a hamiltonian path, followed by moving disc $n+1$ to peg 1.

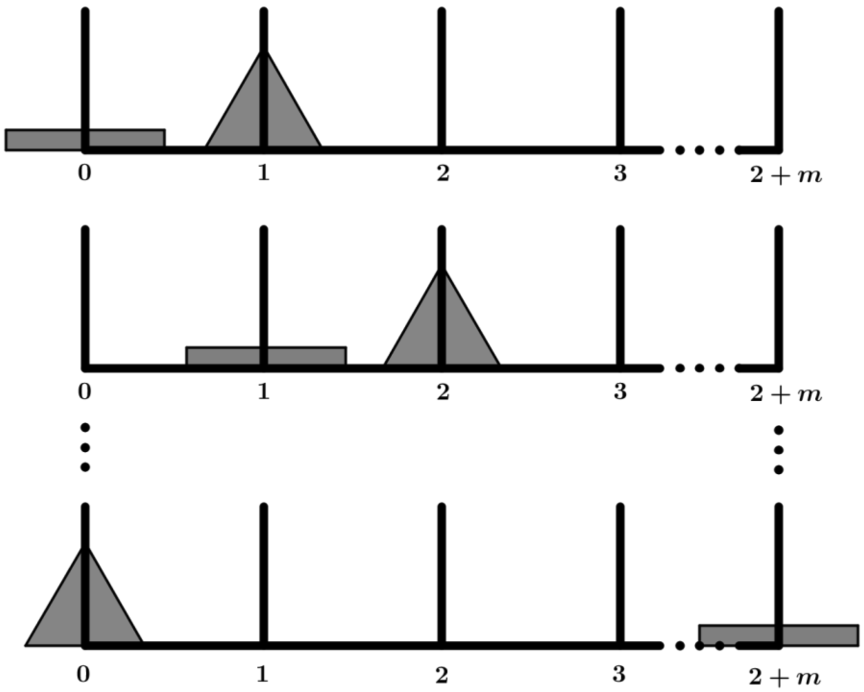
In this step, we've gone through every vertex with a 0 in the last entry, ending on vertex $(2,2,\ldots,2,1)$.



Theorem 1, Part II

Continuing in this way, we transfer the n-tower through a hamiltonian path from peg $i + 1$ to peg $i + 2$ for each $i \in \{0,1, \dots, 2 + m\}$, following each by a single move of disc $n + 1$ from peg i to peg $i + 1$, where each step is modulo $3 + m$.

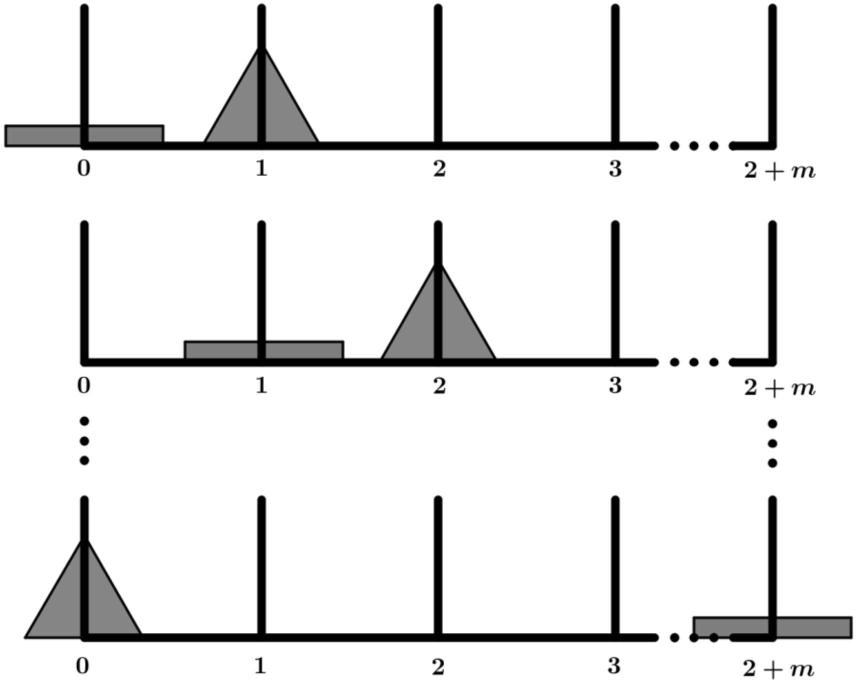
In each step, we go through every vertex with an i in the last entry.



Theorem 1, Part II

The process terminates when we transfer the n-tower back to peg 1, followed by moving disc $n + 1$ to peg 0.

We have completed a path in H_m^{n+1} that goes through every vertex exactly once and ends on the initial vertex. Thus H_m^{n+1} contains a hamiltonian cycle. ■



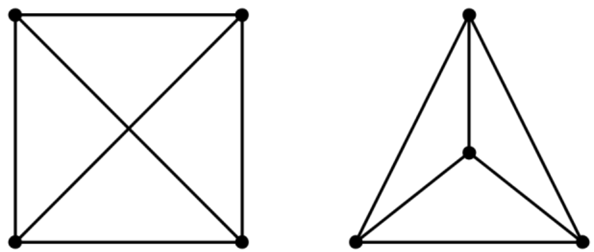
Planar Graphs

Definition

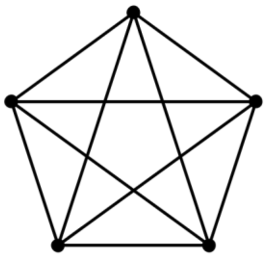
A graph G is called planar if it can be drawn in the plane without any crossings.

Example:

The complete graph K_4 is planar.



The complete graph K_5 is not planar



Planarity of H_m^n

Theorem 2

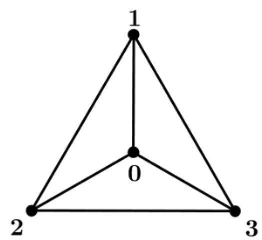
The only planar Hanoi graphs are H_0^n, H_1^1, and H_1^2.

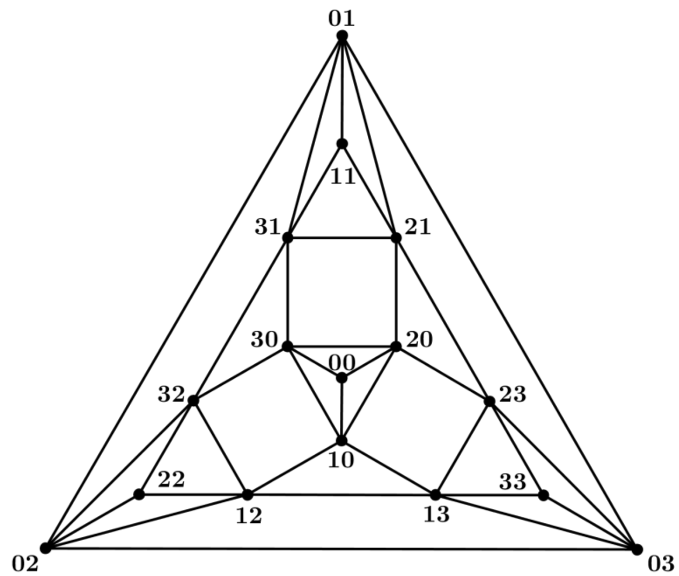
Proof:

- Part I: We will show that H_1^1 and H_1^2 are planar by constructing planar embeddings of each.
- Part II: We will show by induction that H_0^n is planar for all $n \in \mathbb{N}$.
- Part III: We will show that H_m^n is non-planar for all $m \geq 2$ and $n \geq 1$.
- Part IV: We will show that H_1^n is non-planar for all $n \geq 3$.

Theorem 2, Part I

H_1^1 and H_1^2 are planar, as demonstrated by planar embeddings.





Note that, since H_1^2 is 3-connected (there is no pair of vertices whose deletion results in a disconnected graph), this planar embedding of H_1^2 is essentially unique.

Theorem 2, Part I

m, n	1	2	3	4	5	...
0						
1	Y	Y				
2						
3						
4						
5						
⋮						

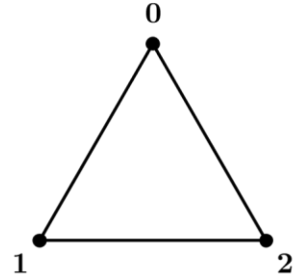
Theorem 2, Part II

We will show by induction on n that H_0^n allows a planar embedding, whose infinite face is the complement of an equilateral triangle with side length $2^n - 1$, and whose corners are the perfect states.

Base Case: Let $n = 1$.

The graph H_0^1 corresponds to the Tower of Hanoi puzzle with 1 disc on 3 pegs. The disc can move freely between the pegs, so H_0^1 is isomorphic to the complete graph K_3.

Thus H_0^1 is planar and it can be drawn as an equilateral triangle with side length $1 = 2^1 - 1$.

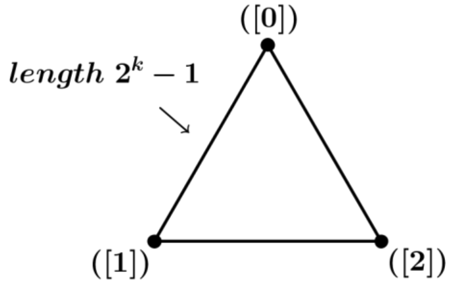


Theorem 2, Part II

Induction Hypothesis:

Fix any $k \in \mathbb{N}$ and suppose H_0^k can be drawn without crossings such that its infinite face is the complement of an equilateral triangle with side length $2^k - 1$ and the corners are the perfect states.

Label the perfect states of H_0^k by $([0])$, $([1])$, and $([2])$, where $([i])$ is the k-tuple consisting of all i's.



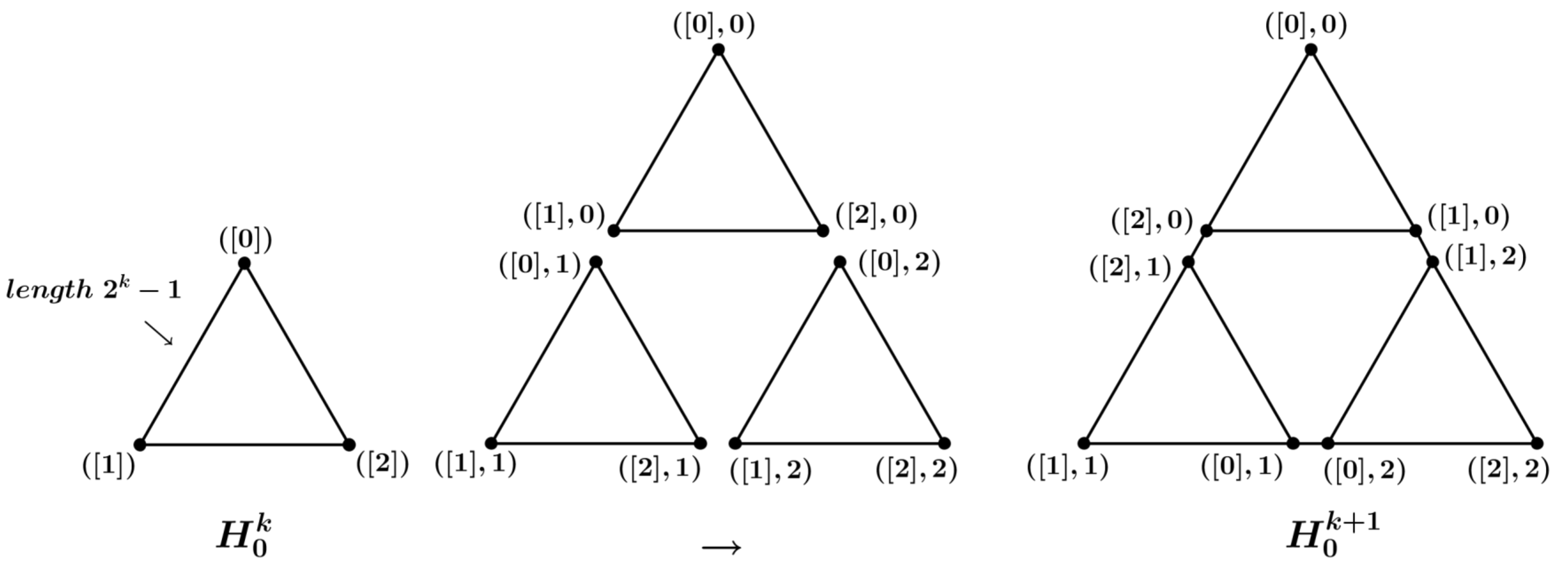
Theorem 2, Part II

We construct H_0^{k+1} in the following way.

- Take 3 copies of H_0^k, one for each possible position of disc $k+1$ (peg 0, 1, or 2).
- Relabel their vertices with $(k+1)$-tuples ending in 0, 1, and 2, respectively.
- Add 3 edges to form the adjacencies $([0], 1) \sim ([0], 2)$, $([1], 0) \sim ([1], 2)$, and $([2], 0) \sim ([2], 1)$.
- Since each of the 3 copies of H_0^k is an equilateral triangle, through flips we can arrange them so that each of the three edges added are the middle edges of a new equilateral triangle with side length

$$2\left(2^k - 1\right) + 1 = 2^{k+1} - 1.$$

Theorem 2, Part II



Theorem 2, Part II

We certainly have the adjacencies $([0],1)\sim([0],2)$, $([1],0)\sim([1],2)$, and $([2],0)\sim([2],1)$ in H_0^{k+1}, since if the k-tower of smaller discs are all on one peg, then disc $k+1$ is free to move between the other two pegs.

To verify that exactly 3 edges are added to the 3 copies of H_0^k to form H_0^{k+1}, we can use the edge count formula for H_m^n

$$|E_m^n| = \frac{(3+m)(2+m)}{4}\left[(3+m)^n - (1+m)^n\right]$$

to show that

$$\left|E_0^{k+1}\right| = 3\left|E_0^k\right| + 3.$$

Thus H_0^n is planar for all $n \in \mathbb{N}$.

Theorem 2, Part II

m, n	1	2	3	4	5	…
0	**Y**	**Y**	**Y**	**Y**	**Y**	…
1	Y	Y				
2						
3						
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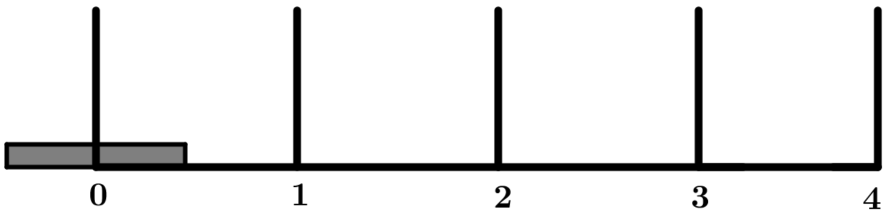
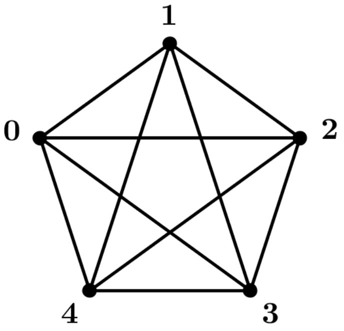
⋮						

Theorem 2, Part III

The Hanoi graph H_2^1 is isomorphic to the complete graph K_5, which is nonplanar.





For any $m \geq 2$ and $n \geq 1$, the Tower of Hanoi puzzle has at least 5 pegs.

In any regular state, the smallest disc can move freely between any set of 5 pegs, so K_5 is a subgraph of the corresponding Hanoi graph.

Thus H_m^n is non-planar for all $m \geq 2$ and $n \geq 1$.

Theorem 2, Part III

m, n	1	2	3	4	5	…
0	Y	Y	Y	Y	Y	…
1	Y	Y				
2	**N**	**N**	**N**	**N**	**N**	…
3	**N**	**N**	**N**	**N**	**N**	…
4	**N**	**N**	**N**	**N**	**N**	…
5	**N**	**N**	**N**	**N**	**N**	…
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem 2, Part IV

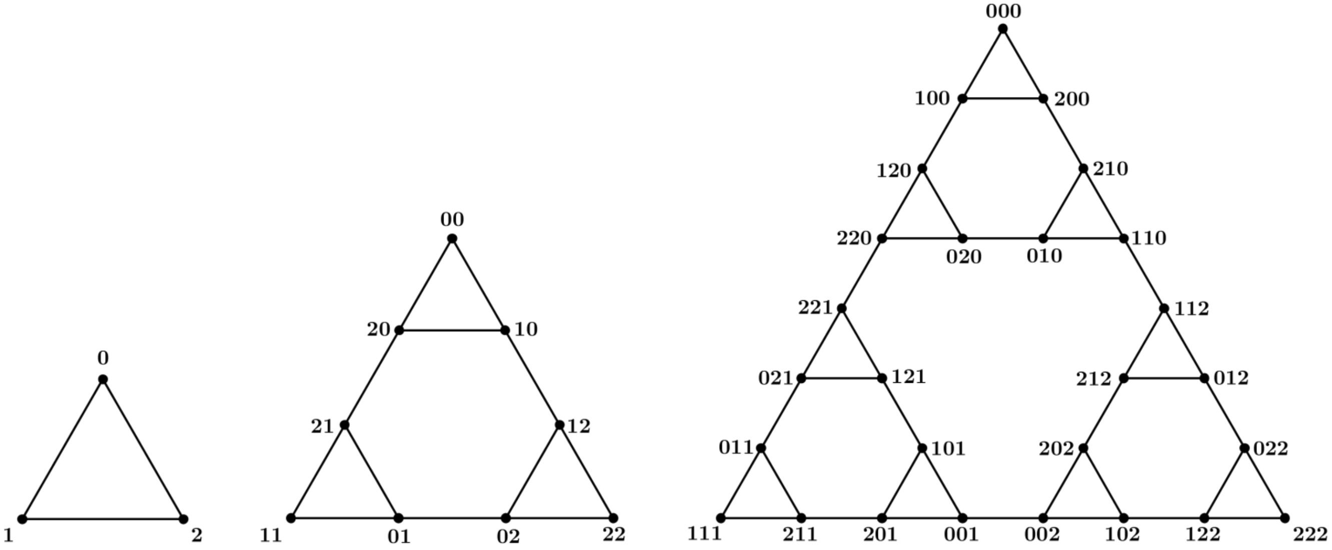
Lemma 2

Fix any $m, n \in \mathbb{N}$, any $k \in \mathbb{N}$ such that $k < n$.

Fix any $l \in \{0,1,\dots,2+m\}$.

Let $S = \{(x_1, x_2, \dots, x_n) | x_{k+1} = x_{k+2} = \cdots = x_n = l\}$.

Then the subgraph of H_m^n induced by S is isomorphic to H_m^k.



Theorem 2, Part IV

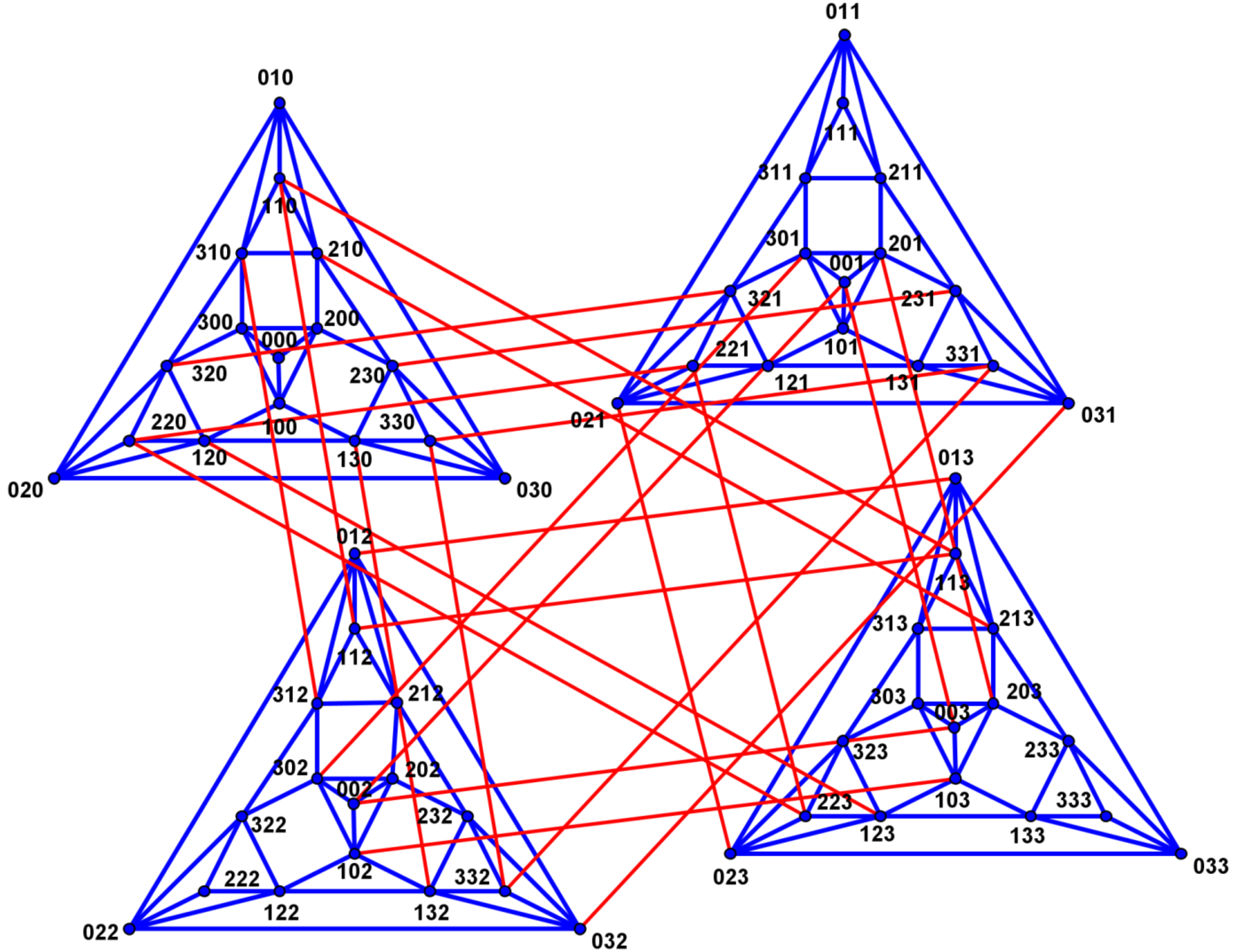
By Lemma 2, H_1^3 is a subgraph of H_1^n for all $n > 3$.
So we need only show that H_1^3 is non-planar.

Kuratowski's Theorem:
If a graph G contains a subgraph that is a K_5 or $K_{3,3}$ subdivision, then G is non-planar.

We can construct H_1^3 by taking 4 copies of H_1^2, one for each position of the largest disc, and adding 24 edges corresponding to legal moves of the largest disc.

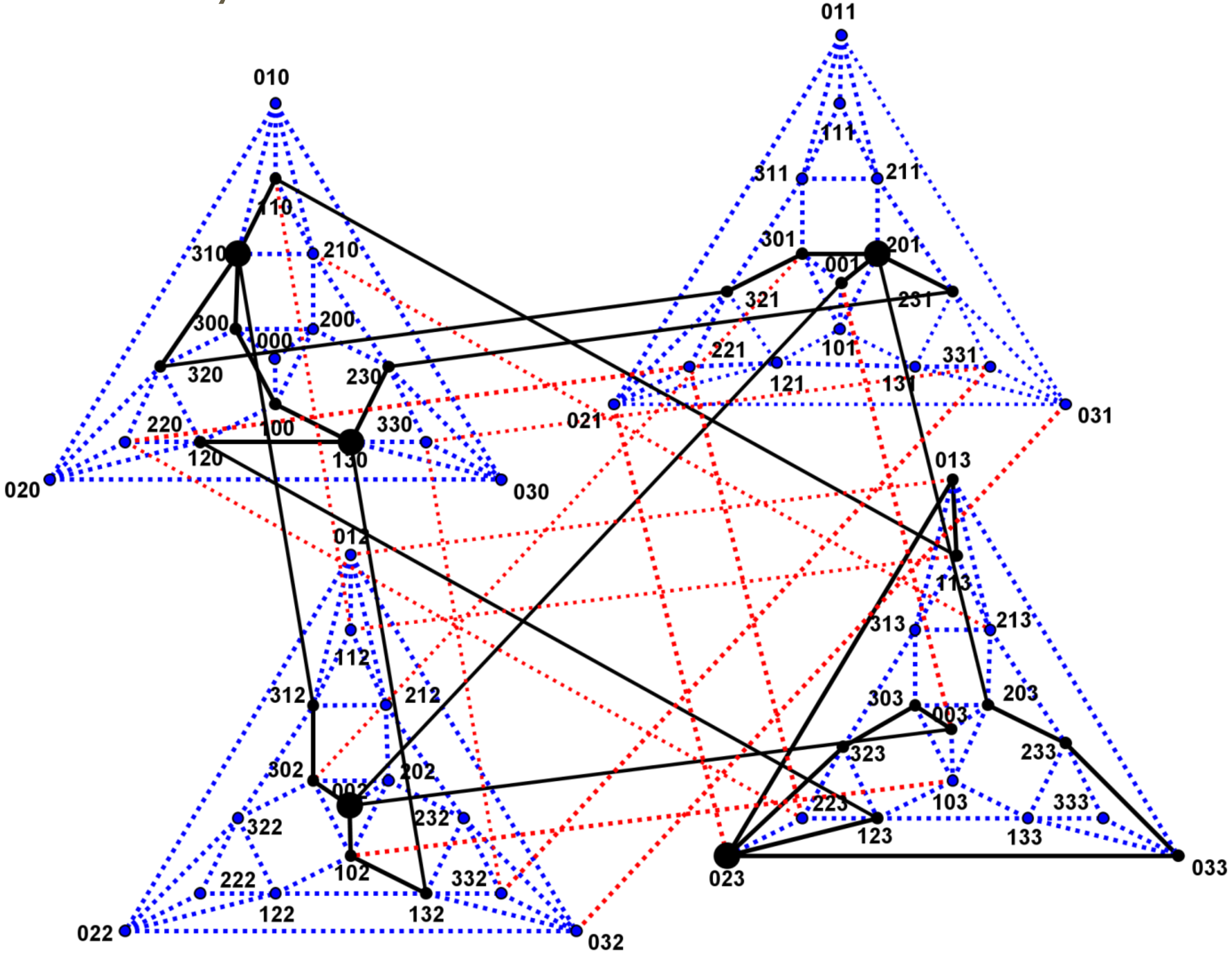
Theorem 2, Part IV

H_1^3



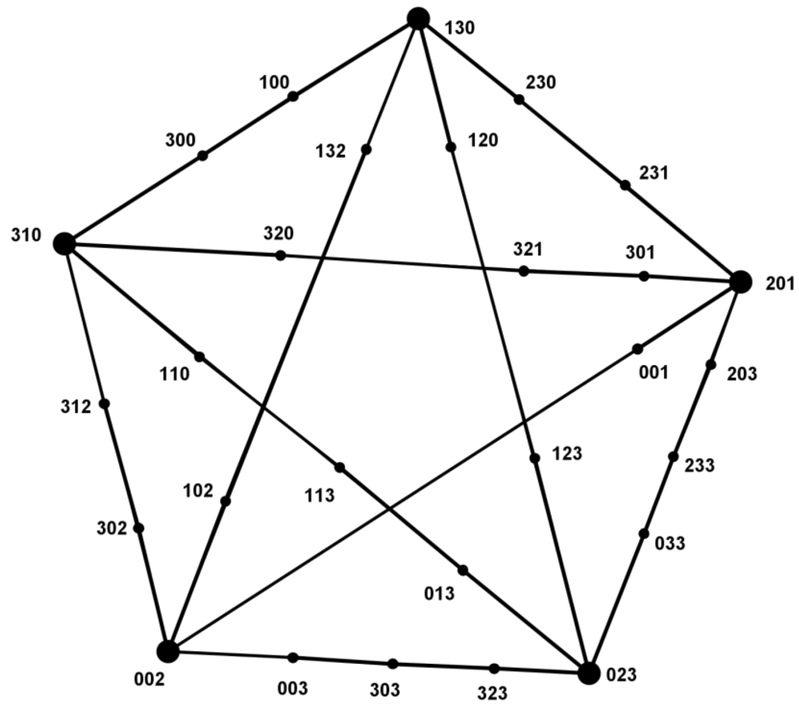
Theorem 2, Part IV

H_1^3



Theorem 2, Part IV

K_5 subdivision subgraph of H_1^3:



Thus H_1^n is non-planar for all $n \geq 3$. ■

Theorem 2, Part IV

m, n	1	2	3	4	5	…
0	Y	Y	Y	Y	Y	…
1	Y	Y	**N**	**N**	**N**	…
2	N	N	N	N	N	…
3	N	N	N	N	N	…
4	N	N	N	N	N	…
5	N	N	N	N	N	…
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Conclusion

Planarity:

The only planar Hanoi graphs are H_0^n, H_1^1, and H_1^2.

Hamiltonicity:

Every Hanoi graph H_m^n is hamiltonian.

Some Open Problems

- The Frame-Stewart Conjecture for more than 4 pegs.
- The genera and crossing numbers for non-planar Hanoi graphs.
- A formula for the average distance in H_m^n for $m \geq 1$.
- The diameter of H_m^n for $m \geq 1$.

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